

Chiral perturbation theory with virtual photons and leptons^{*}

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Abstract. We construct a low-energy effective field theory which allows for a full treatment of the isospin-breaking effects in semileptonic weak interactions. To this end, we enlarge the particle spectrum of chiral perturbation theory with virtual photons by including also the light leptons as dynamical degrees of freedom. Using super-heat-kernel techniques, we determine the additional one-loop divergences generated by the presence of virtual leptons and give the full list of associated local counterterms. We illustrate the use of our effective theory by applying it to the decays $\pi \rightarrow \ell\nu_\ell$ and $K \rightarrow \ell\nu_\ell$.

1 Introduction

In the last few years, the predictions of chiral perturbation theory [1–3] have been carried forward to a remarkable degree of accuracy. In particular, the result for the pion–pion scattering amplitude is now available at the two-loop level [4] in the standard case. Let us illustrate this theoretical progress by the following numbers: Weinberg’s calculation [5] of the scattering amplitude at leading order in the low-energy expansion gives the value $a_0^0 = 0.16$ for the isospin zero S wave scattering length. The one-loop calculation [6] shifts the leading order term to $a_0^0 = 0.20$. Finally, the recent analysis to order p^6 [4] predicts the scattering length to lie within the range $0.206 \leq a_0^0 \leq 0.217$ (see also [7]). An analogous calculation at next-to-next-to-leading order has also been performed in the framework of generalized chiral perturbation theory [8], which allows also for smaller values of the quark condensate than the standard scheme. In the generalized framework, an accurate determination of a_0^0 within the whole range of the present experimental value, $a_0^0 = 0.26 \pm 0.05$ [9], can be interpreted in terms of the size of $\langle 0|\bar{q}q|0\rangle$, whereas standard chiral perturbation theory, resting on the assumption that $\langle 0|\bar{q}q|0\rangle$ is large, gives a rather precise prediction for this scattering length. New high statistics data which are expected for the near future may allow one to determine the nature of chiral symmetry breaking and to decide about the validity of the standard chiral expansion scheme. In this context, the measurement of the lifetime of $\pi^+\pi^-$ atoms at CERN [10], or new high statistics $K_{\ell 4}$ experiments by the E865 and KLOE collaborations at BNL [11] and DAΦNE [12], respectively, are of particular interest.

The theoretical results mentioned above were obtained by neglecting all isospin-breaking effects, i.e. in the limit $m_u = m_d$, $e = 0$. However, once even two-loop effects are taken into account, such an approach is not sufficient any more. It has been shown explicitly [13] that the electromagnetic corrections to the S wave scattering lengths are of comparable size to the $\mathcal{O}(p^6)$ strong interaction contributions. (See also [14] for a discussion of electromagnetic effects in neutral pion scattering.) Such an analysis requires, of course, an extension of the usual low-energy effective theory. While isospin-breaking effects generated by a non-vanishing quark mass difference $m_d - m_u$ are fully contained in the pure QCD sector of the effective chiral Lagrangian, the treatment of isospin violation of electromagnetic origin requires the inclusion of virtual photons and the appropriate local terms up to $\mathcal{O}(e^2p^2)$. The suitable theoretical framework has been worked out in [15–17] for the three flavour case and in [14,13] for chiral SU(2). With these theoretical tools, it is possible to obtain the chiral structure of the electromagnetic contributions to $\mathcal{O}(e^2p^2)$ for all purely mesonic matrix elements.

The analysis of electromagnetic corrections in semileptonic reactions requires still a further extension of chiral perturbation theory: In this case, also the light leptons have to be included as explicit dynamical degrees of freedom. (An older discussion of radiative corrections in semileptonic weak interactions, with a current algebra treatment of the hadronic matrix elements involved, can be found in [18].) Only within such a framework, one will have full control over all possible isospin-breaking effects in the analysis of high statistics $K_{\ell 4}$ data which will constitute an important source of information on the $\pi-\pi$ scattering parameters and low-energy phase shifts. The same refined methods are, of course, also necessary for the interpretation of forthcoming high precision experiments on other semileptonic decays like $K_{\ell 3}$, etc.

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It is the purpose of this paper to lay the necessary theoretical foundations for the full treatment of electromagnetic effects in the semileptonic processes of the pseudoscalar octet within the framework of an effective low-energy theory. In Sect. 2 we construct our lowest-order effective Lagrangian and we define our (extended) chiral counting rules. The additional one-loop divergences generated by the presence of virtual leptons are obtained in Sect. 3 by using recently developed super-heat-kernel methods [19,20]. The full list of local counterterms arising at next-to-leading order is presented in Sect. 4. In Sect. 5 we apply our effective theory to the decays $\pi \rightarrow \ell\nu_\ell$ and $K \rightarrow \ell\nu_\ell$. Our conclusions, together with an outlook to possible applications and extensions of the present work, are summarized in Sect. 6. Several expressions which would only interrupt the argument in the text are collected in the Appendix.

2 The effective Lagrangian to lowest order

For a complete treatment of the electromagnetic effects in the semileptonic decays of pions and kaons, not only the pseudoscalars but also the photon and the light leptons have to be included as dynamical degrees of freedom in an appropriate effective Lagrangian. Its construction starts with QCD in the limit $m_u = m_d = m_s = 0$. The resulting symmetry under the chiral group $G = SU(3)_L \times SU(3)_R$ is spontaneously broken to $SU(3)_V$. The pseudoscalar mesons (π, K, η) are interpreted as the corresponding Goldstone fields φ_i ($i = 1, \dots, 8$) acting as coordinates of the coset space $SU(3)_L \times SU(3)_R / SU(3)_V$. The coset variables $u_{L,R}(\varphi)$ transform as

$$\begin{aligned} u_L(\varphi) &\xrightarrow{G} g_L u_L h(g, \varphi)^{-1}, \\ u_R(\varphi) &\xrightarrow{G} g_R u_R h(g, \varphi)^{-1}, \\ g &= (g_L, g_R) \in SU(3)_L \times SU(3)_R, \end{aligned} \quad (2.1)$$

where $h(g, \varphi)$ is the nonlinear realization of G [21].

The photon field A_μ and the leptons ℓ, ν_ℓ ($\ell = e, \mu$) are introduced in

$$u_\mu = i[u_R^\dagger(\partial_\mu - ir_\mu)u_R - u_L^\dagger(\partial_\mu - il_\mu)u_L] \quad (2.2)$$

by adding appropriate terms to the usual external vector and axial-vector sources v_μ, a_μ . At the quark level, this procedure corresponds to the usual minimal coupling prescription in the case of electromagnetism, and to Cabibbo universality in the case of the charged weak currents:

$$\begin{aligned} l_\mu &= v_\mu - a_\mu - eQ_L^{\text{em}} A_\mu + \sum_\ell (\bar{\ell} \gamma_\mu \nu_{\ell L} Q_L^{\text{w}} + \bar{\nu}_{\ell L} \gamma_\mu \ell Q_L^{\text{w}\dagger}), \\ r_\mu &= v_\mu + a_\mu - eQ_R^{\text{em}} A_\mu. \end{aligned} \quad (2.3)$$

The 3×3 matrices $Q_{L,R}^{\text{em}}, Q_L^{\text{w}}$ are spurion fields with the transformation properties

$$Q_L^{\text{em,w}} \xrightarrow{G} g_L Q_L^{\text{em,w}} g_L^\dagger, \quad Q_R^{\text{em}} \xrightarrow{G} g_R Q_R^{\text{em}} g_R^\dagger \quad (2.4)$$

under the chiral group. We also define

$$Q_L^{\text{em,w}} := u_L^\dagger Q_L^{\text{em,w}} u_L, \quad Q_R^{\text{em}} := u_R^\dagger Q_R^{\text{em}} u_R \quad (2.5)$$

transforming as

$$\begin{aligned} Q_L^{\text{em,w}} &\xrightarrow{G} h(g, \varphi) Q_L^{\text{em,w}} h(g, \varphi)^{-1}, \\ Q_R^{\text{em}} &\xrightarrow{G} h(g, \varphi) Q_R^{\text{em}} h(g, \varphi)^{-1}. \end{aligned} \quad (2.6)$$

In the end, one identifies $Q_{L,R}^{\text{em}}$ with the quark charge matrix

$$Q^{\text{em}} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}, \quad (2.7)$$

whereas the weak spurion is taken at

$$Q_L^{\text{w}} = -2\sqrt{2} G_F \begin{bmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.8)$$

where G_F is the Fermi coupling constant and V_{ud}, V_{us} are Kobayashi–Maskawa matrix elements.

With these building blocks, our lowest-order effective Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + e^2 F_0^4 Z \langle Q_L^{\text{em}} Q_R^{\text{em}} \rangle \\ &\quad - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_\ell [\bar{\ell}(i\partial + eA - m_\ell)\ell + \bar{\nu}_{\ell L} i\partial \nu_{\ell L}], \end{aligned} \quad (2.9)$$

where $\langle \rangle$ denotes the trace in three-dimensional flavor space. F_0 is the pion decay constant in the chiral limit and in the absence of electroweak interactions. Explicit chiral symmetry breaking is included in $\chi_+ = u_L^\dagger \chi u_L + u_R^\dagger \chi u_R$ by the substitution $\chi \rightarrow 2B_0 \mathcal{M}_{\text{quark}}$, where B_0 is related to the quark condensate in the chiral limit by $\langle 0|\bar{q}q|0\rangle = -F_0^2 B_0$. The numerical value of the coupling constant Z can be determined from the mass difference of the charged and the neutral pion. The relation $M_{\pi^\pm}^2 - M_{\pi^0}^2 = 2e^2 Z F_0^2$ resulting from (2.9) implies $Z \simeq 0.8$.

We adopt an expansion scheme where the electric charge e , the lepton masses m_e, m_μ and fermion bilinears are considered as quantities of order p in the chiral counting, where p is a typical meson momentum. Note, however, that terms of $\mathcal{O}(e^4)$ will be neglected throughout.

3 One-loop divergences

For the construction of the one-loop functional we first add a gauge-breaking term (we are using the Feynman gauge) and external sources to (2.9):

$$\begin{aligned} \mathcal{L}_{\text{eff}} &\rightarrow \mathcal{L}_{\text{eff}} - \frac{1}{2} (\partial_\mu A^\mu)^2 - J_\mu A^\mu \\ &\quad + \sum_\ell (\bar{\rho}_\ell \ell + \bar{\ell} \rho_\ell + \bar{\sigma}_\ell \nu_{\ell L} + \bar{\nu}_{\ell L} \sigma_\ell). \end{aligned} \quad (3.1)$$

Then we expand the lowest-order action associated with (3.1) around the solutions $\varphi_{\text{cl}}, A_{\text{cl}}^\mu, \ell_{\text{cl}}, \nu_{\ell L, \text{cl}}$ of the classical equations of motion. In the standard ‘‘gauge’’ $u_{\text{R}}(\varphi_{\text{cl}}) = u_{\text{L}}(\varphi_{\text{cl}})^\dagger =: u(\varphi_{\text{cl}})$ a convenient choice of the pseudoscalar fluctuation variables ξ_i ($i = 1, \dots, 8$) is given by

$$\begin{aligned} u_{\text{R}} &= u_{\text{cl}} e^{i\xi_i \lambda_i / 2F_0}, & u_{\text{L}} &= u_{\text{cl}}^\dagger e^{-i\xi_i \lambda_i / 2F_0}, \\ \xi_i(\varphi_{\text{cl}}) &= 0, \end{aligned} \quad (3.2)$$

with the Gell-Mann matrices λ_i ($i = 1, \dots, 8$). For the photon and the fermions we write

$$A^\mu = A_{\text{cl}}^\mu + \varepsilon^\mu, \quad \ell = \ell_{\text{cl}} + \eta_\ell, \quad \nu_{\ell L} = \nu_{\ell L, \text{cl}} + \zeta_{\ell L}. \quad (3.3)$$

In the following formulas, we shall drop the subscript ‘‘cl’’ for simplicity. The classical equations of motion take the form

$$\begin{aligned} \nabla_\mu u^\mu &= \frac{i}{2} \left(\chi_- - \frac{1}{3} \langle \chi_- \rangle \right) \\ &\quad + 2ie^2 F_0^2 Z [\mathcal{Q}_{\text{R}}^{\text{em}}, \mathcal{Q}_{\text{L}}^{\text{em}}], \\ \square A_\mu &= J_\mu + \frac{eF_0^2}{2} \langle u_\mu (\mathcal{Q}_{\text{R}}^{\text{em}} - \mathcal{Q}_{\text{L}}^{\text{em}}) \rangle \\ &\quad - e \sum_\ell \bar{\ell} \gamma_\mu \ell, \\ (i \not{\partial} + e \not{A} - m_\ell) \ell &= -\rho_\ell + \frac{F_0^2}{2} \langle u_\mu \mathcal{Q}_{\text{L}}^{\text{w}} \rangle \gamma^\mu \nu_{\ell L}, \\ i \not{\partial} \nu_{\ell L} &= -\sigma_\ell + \frac{F_0^2}{2} \langle u_\mu \mathcal{Q}_{\text{L}}^{\text{w}\dagger} \rangle \gamma^\mu \ell_L, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \nabla_\mu &= \partial_\mu + [\Gamma_\mu, \quad], \\ \Gamma_\mu &= \frac{1}{2} [u^\dagger (\partial_\mu - i r_\mu) u + u (\partial_\mu - i l_\mu) u^\dagger], \\ \chi_- &= u^\dagger \chi u^\dagger - u \chi^\dagger u. \end{aligned} \quad (3.5)$$

The solutions of (3.4) are uniquely determined functionals of the external sources $v_\mu, a_\mu, \chi, J_\mu, \rho_\ell, \sigma_\ell$. (Note that the usual Feynman boundary conditions are always implicitly understood.)

Expanding (3.1) up to terms quadratic in the variables $\xi_i, \varepsilon_\mu, \eta_\ell, \zeta_{\ell L}$, we obtain the second-order fluctuation Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)} &= -\frac{1}{2} \xi_i (d \cdot d + s)_{ij} \xi_j \\ &\quad + \frac{1}{2} \varepsilon_\mu \square \varepsilon^\mu + \frac{e^2 F_0^2}{4} \langle (\mathcal{Q}_{\text{R}}^{\text{em}} - \mathcal{Q}_{\text{L}}^{\text{em}})^2 \rangle \varepsilon_\mu \varepsilon^\mu \\ &\quad - \frac{ieF_0}{4} \langle [u_\mu, \mathcal{Q}_{\text{R}}^{\text{em}} + \mathcal{Q}_{\text{L}}^{\text{em}}] \lambda_i \rangle \xi_i \varepsilon^\mu \\ &\quad + \frac{eF_0}{2} \langle (\mathcal{Q}_{\text{R}}^{\text{em}} - \mathcal{Q}_{\text{L}}^{\text{em}}) \lambda_i \rangle (d_{ij}^\mu \xi_j) \varepsilon_\mu \\ &\quad + \sum_\ell \{ \bar{\eta}_\ell (i \not{\partial} + e \not{A} - m_\ell) \eta_\ell + \overline{\zeta_{\ell L}} i \not{\partial} \zeta_{\ell L} \\ &\quad + \frac{iF_0}{4} \langle [u_\mu, (\bar{\ell} \gamma^\mu \zeta_{\ell L} + \bar{\eta}_\ell \gamma^\mu \nu_{\ell L}) \mathcal{Q}_{\text{L}}^{\text{w}} + \text{h.c.}] \lambda_i \rangle \xi_i \end{aligned}$$

$$\begin{aligned} &+ \frac{F_0}{2} \langle [(\bar{\ell} \gamma_\mu \zeta_{\ell L} + \bar{\eta}_\ell \gamma_\mu \nu_{\ell L}) \mathcal{Q}_{\text{L}}^{\text{w}} + \text{h.c.}] \lambda_i \rangle d_{ij}^\mu \xi_j \\ &+ \frac{eF_0^2}{2} \varepsilon_\mu \langle (\mathcal{Q}_{\text{R}}^{\text{em}} - \mathcal{Q}_{\text{L}}^{\text{em}}) [(\bar{\ell} \gamma^\mu \zeta_{\ell L} + \bar{\eta}_\ell \gamma^\mu \nu_{\ell L}) \mathcal{Q}_{\text{L}}^{\text{w}} \\ &+ \text{h.c.}] \rangle + e(\bar{\eta}_\ell \not{\partial} \ell + \bar{\ell} \not{\partial} \eta_\ell) \\ &- \frac{F_0^2}{2} \langle u_\mu (\bar{\eta}_\ell \gamma^\mu \zeta_{\ell L} \mathcal{Q}_{\text{L}}^{\text{w}} + \text{h.c.}) \rangle, \end{aligned} \quad (3.6)$$

where

$$d_{ij}^\mu = \delta_{ij} \partial^\mu - \frac{1}{2} \langle \Gamma^\mu [\lambda_i, \lambda_j] \rangle, \quad (3.7)$$

and

$$\begin{aligned} s_{ij} &= \langle \frac{1}{8} (u \cdot u + \chi_+) \{ \lambda_i, \lambda_j \} - \frac{1}{4} u_\mu \lambda_i u^\mu \lambda_j \\ &\quad + \frac{e^2 F_0^2 Z}{2} \{ \mathcal{Q}_{\text{R}}^{\text{em}}, \mathcal{Q}_{\text{L}}^{\text{em}} \} \{ \lambda_i, \lambda_j \} \\ &\quad - e^2 F_0^2 Z (\lambda_i \mathcal{Q}_{\text{R}}^{\text{em}} \lambda_j \mathcal{Q}_{\text{L}}^{\text{em}} + \lambda_j \mathcal{Q}_{\text{R}}^{\text{em}} \lambda_i \mathcal{Q}_{\text{L}}^{\text{em}}) \rangle. \end{aligned} \quad (3.8)$$

For the following intermediate steps it is convenient to switch to Euclidean space. The second-order fluctuation Lagrangian can then be written in the form

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{1}{2} \Phi^T (-D \cdot D + Y) \Phi + \bar{\Psi} (\not{D} + m) \Psi \\ &\quad + i \bar{\Psi} \gamma_\rho (\beta D_\rho + \alpha_\rho) \Phi \\ &\quad + i \Phi^T (-\overleftarrow{D}_\rho \bar{\beta} + \bar{\alpha}_\rho) \gamma_\rho \Psi. \end{aligned} \quad (3.9)$$

In (3.9) all bosonic fluctuation variables are collected in a multicomponent field

$$\Phi = \begin{bmatrix} \xi_i \\ \varepsilon_\mu \end{bmatrix}, \quad (3.10)$$

where D_ρ denotes an associated covariant derivative:

$$D_\rho = \partial_\rho + X_\rho. \quad (3.11)$$

Analogously, the fermionic fluctuation fields are combined in

$$\Psi = \begin{bmatrix} \eta_a \\ \zeta_{mL} \end{bmatrix}, \quad (3.12)$$

with a covariant derivative

$$\mathcal{D}_\rho = \partial_\rho + i f_\rho. \quad (3.13)$$

The explicit expressions for the matrix fields $X_\rho, Y, f_\rho, m, \beta, \alpha_\rho$ are given in the Appendix.

In the notation of [19], the second-order fluctuation action associated with (3.9) has the general structure

$$S^{(2)} = \frac{1}{2} \Phi^T \mathcal{A} \Phi + \bar{\Psi} \mathcal{B} \Psi + \Phi^T \overleftarrow{\Gamma} \Psi + \bar{\Psi} \Gamma \Phi. \quad (3.14)$$

The one-loop functional $W_{L=1}$ is given by the Gaussian functional integral

$$e^{-W_{L=1}} = \int [d\Phi d\Psi d\bar{\Psi}] e^{-S^{(2)}}, \quad (3.15)$$

leading to

$$W_{L=1} = \frac{1}{2} \text{Tr} \ln \frac{\mathcal{A}}{\mathcal{A}_0} - \text{Tr} \ln \frac{\mathcal{B}}{\mathcal{B}_0} \quad (3.16)$$

$$- \sum_{n=1}^{\infty} \frac{1}{2n} \text{Tr} \left(\mathcal{A}^{-1} \bar{\Gamma} \mathcal{B}^{-1} \Gamma - \mathcal{A}^{-1} \Gamma^T \mathcal{B}^{-1T} \bar{\Gamma}^T \right)^n,$$

where $\mathcal{A}_0, \mathcal{B}_0$ denote the free-field limit of \mathcal{A} and \mathcal{B} , respectively.

The term $(1/2) \text{Tr} \ln(\mathcal{A}/\mathcal{A}_0)$ corresponds to purely bosonic loops (only pseudoscalar and/or photon propagators in the loop); its divergent part has been calculated by Gasser and Leutwyler [3] for the strong sector and by Urech [15] for the electromagnetic sector. In our theory, the divergences generated by the bosonic loops have exactly the same form, however with the generalized source term l_μ defined in (2.3). As a consequence, also contributions with external lepton fields are induced in this way.

Purely fermionic loops are described by $-\text{Tr} \ln(\mathcal{B}/\mathcal{B}_0)$. The divergent part of this term affects only the wave function renormalization of the photon, as contributions of order $G_{\mathbb{F}}^2$ will be neglected.

The last term in (3.16) is a genuinely new part occurring in our effective theory with virtual leptons. It describes all one-loop graphs where boson and fermion propagators alternate. In order to determine the ultraviolet divergences associated with this mixed one-loop functional, we employ the super-heat-kernel technique developed in [19]. To this end we introduce the supermatrix

$$K = \begin{bmatrix} \mathcal{A} & \sqrt{2\mu}\bar{\Gamma} \\ \sqrt{2\mu}\Gamma & \mu\mathcal{B} \end{bmatrix}, \quad (3.17)$$

where μ is an arbitrary mass parameter. The one-loop functional can now be written as

$$W_{L=1} = \frac{1}{2} \text{Str} \ln \frac{K}{K_0} - \frac{1}{2} \text{Tr} \ln \frac{\mathcal{B}}{\mathcal{B}_0} + \dots \quad (3.18)$$

The dots refer to terms at least quartic in the fermion fields, which are irrelevant for our present purposes. In the proper-time formulation, the first term in (3.18) assumes the form

$$\frac{1}{2} \text{Str} \ln \frac{K}{K_0} = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{Str} (e^{-\tau K} - e^{-\tau K_0}) \quad (3.19)$$

$$= -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^d x \text{str} \langle x | e^{-\tau K} - e^{-\tau K_0} | x \rangle.$$

The further evaluation of this expression is considerably simplified [20] by the observation that the action associated with (3.9) is invariant under local gauge transformations

$$\begin{aligned} \Phi(x) &\rightarrow R(x)\Phi(x), & R(x)^T R(x) &= \mathbf{1}, \\ \Psi(x) &\rightarrow U(x)\Psi(x), & U(x)^\dagger U(x) &= \mathbf{1}, \\ X_\mu &\rightarrow R\partial_\mu R^{-1} + R X_\mu R^{-1}, \\ Y &\rightarrow R Y R^{-1}, \\ i f_\mu &\rightarrow U\partial_\mu U^{-1} + U i f_\mu U^{-1}, \\ \alpha_\mu &\rightarrow U\alpha_\mu R^{-1}, \\ \beta &\rightarrow U\beta R^{-1}, \\ m &\rightarrow U m U^{-1}. \end{aligned} \quad (3.20)$$

Consequently, also the divergent part of the one-loop functional exhibits this symmetry property [22]. The matrix fields Y, α_μ, β, m together with their covariant derivatives

$$\begin{aligned} \nabla_\mu Y &:= \partial_\mu Y + [X_\mu, Y], \\ \nabla_\mu \alpha_\nu &:= \partial_\mu \alpha_\nu + i f_\mu \alpha_\nu - \alpha_\nu X_\mu, \\ \nabla_\mu \beta &:= \partial_\mu \beta + i f_\mu \beta - \beta X_\mu, \\ \nabla_\mu m &:= \partial_\mu m + i [f_\mu, m], \end{aligned} \quad (3.21)$$

and the associated ‘‘field-strength’’ tensors

$$\begin{aligned} X_{\mu\nu} &:= \partial_\mu X_\nu - \partial_\nu X_\mu + [X_\mu, X_\nu], \\ f_{\mu\nu} &:= \partial_\mu f_\nu - \partial_\nu f_\mu + i [f_\mu, f_\nu] \end{aligned} \quad (3.22)$$

are therefore the appropriate building blocks for the construction of $W_{L=1}^{\text{div}}$. As a consequence, the following intermediate steps in the further computation of (3.19) may be performed with constant fields [23] $X_\mu, Y = X^2, \alpha_\mu, \beta, f_\mu, m$. As the final result for the one-loop divergences has to be gauge invariant, no information is lost and the full expression for the x dependent fields is recovered by the substitutions

$$\begin{aligned} X^2 &\rightarrow Y, \\ [X_\mu, X^2] &\rightarrow \nabla_\mu Y, \\ i f_\mu \alpha_\nu - \alpha_\nu X_\mu &\rightarrow \nabla_\mu \alpha_\nu, \\ i f_\mu \beta - \beta X_\mu &\rightarrow \nabla_\mu \beta, \\ i [f_\mu, m] &\rightarrow \nabla_\mu m, \\ [X_\mu, X_\nu] &\rightarrow X_{\mu\nu}, \\ i [f_\mu, f_\nu] &\rightarrow f_{\mu\nu}. \end{aligned} \quad (3.23)$$

In this approach, the relevant diagonal matrix element in (3.19) assumes the simple form

$$\begin{aligned} \text{str} \langle x | e^{-\tau K} | x \rangle &= \text{str} \int d^d k \langle x | e^{-\tau K} | k \rangle \langle k | x \rangle \\ &= \text{str} \int \frac{d^d k}{(2\pi)^d} e^{-i k \cdot x} e^{-\tau K} e^{i k \cdot x} \\ &= \text{str} \int \frac{d^d k}{(2\pi)^d} e^{M+N}, \end{aligned} \quad (3.24)$$

with

$$M = -\tau \begin{bmatrix} k^2 - 2i k \cdot X & 0 \\ 0 & \mu(i k + i f + m) \end{bmatrix}, \quad (3.25)$$

$$N = -i\tau \sqrt{2\mu} \begin{bmatrix} 0 & -(i k_\mu + X_\mu) \bar{\beta} \gamma_\mu + \bar{\alpha} \cdot \gamma \\ \gamma_\mu \beta (i k_\mu + X_\mu) + \gamma \cdot \alpha & 0 \end{bmatrix}.$$

We are interested only in the part bilinear in the fermionic matrix N . The corresponding piece of the generating functional (3.16) is just

$$W_{L=1}|_{\bar{\Gamma} \dots \Gamma} := -\text{Tr}(\mathcal{A}^{-1} \bar{\Gamma} \mathcal{B}^{-1} \Gamma). \quad (3.26)$$

The appropriate decomposition of the exponential in (3.24) can be performed by using the formula

$$\exp(M + N) = \exp M \text{P}_s \exp \int_0^1 ds \tilde{N}(s), \quad (3.27)$$

with

$$\tilde{N}(s) := e^{-sM} N e^{sM}$$

and

$$\begin{aligned} \text{P}_s \exp \int_0^1 ds \tilde{N}(s) &:= \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \dots \\ &\int_0^{s_{n-1}} ds_n \tilde{N}(s_1) \tilde{N}(s_2) \dots \tilde{N}(s_n). \end{aligned} \quad (3.28)$$

Picking out the part bilinear in N ,

$$\begin{aligned} \text{str } e^{M+N} &= \\ &\int_0^1 ds \int_0^s ds' \text{str} \left[e^{(1-s)M} N e^{(s-s')M} N e^{s'M} \right] + \dots, \end{aligned} \quad (3.29)$$

a few simple manipulations lead to

$$\begin{aligned} \text{str } e^{M+N} &= -2\mu\tau^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 dz \text{tr} \{ (2i\tau z k \cdot X)^n \\ &\times [-(ik_\mu + X_\mu)\bar{\beta} + \bar{\alpha}_\mu] \gamma_\mu \\ &\times \exp[-\tau z k^2 - \tau\mu(1-z)(i\mathbf{k} + i\mathbf{f} + m)] \\ &\times \gamma_\nu [\beta(ik_\nu + X_\nu) + \alpha_\nu] \} + \dots \end{aligned} \quad (3.30)$$

After integration over z , the μ -dependent terms cancel once the proper-time and the momentum-space integrals are applied. The remaining contribution to $W_{L=1}$ assumes the form

$$\begin{aligned} W_{L=1}|_{\bar{\Gamma}\dots\Gamma} &= \int d^d x \sum_{n=0}^{\infty} \int_0^{\infty} \frac{dt}{t} t^{n+3-d} \\ &\times \int \frac{d^d l}{(2\pi)^d} (l^2)^{-n-1} \text{tr} \{ (2il \cdot X)^n \\ &\times [-(il_\mu/t + X_\mu)\bar{\beta} + \bar{\alpha}_\mu] \gamma_\mu \\ &\times \exp(-i\mathbf{l} - i\mathbf{f} - tm) \\ &\times \gamma_\nu [\beta(il_\nu/t + X_\nu) + \alpha_\nu] \}, \end{aligned} \quad (3.31)$$

where a suitable change of the integration variables has been performed. The divergent part (for $d \rightarrow 4$) can now easily be isolated. Note that terms of the form $\bar{\beta} \dots \beta$ are at least quadratic in G_F (see (A.13) and (A.14)) and will thus be discarded. For the further decomposition of $\exp(-i\mathbf{l} - i\mathbf{f} - tm)$ we employ again (3.27) up to second order in $i\mathbf{f} + m$. Then we perform the momentum-space integration using

$$\begin{aligned} \lim_{d \rightarrow 4} \int \frac{d^d l}{(2\pi)^d} \left(\frac{\cos l}{l^2}, \frac{\sin l}{l^3}, \frac{\sin l}{l} \right) &= \frac{1}{(4\pi)^2} (-2, 2, -4), \\ l &:= \sqrt{l_\mu l_\mu}. \end{aligned} \quad (3.32)$$

In the next step, one has to identify the appropriate gauge-invariant combinations (constituting a non-trivial check of our calculation) and reconstruct the full result by using

(3.23). In this way, we obtain

$$\begin{aligned} W_{L=1}^{\text{div}}|_{\bar{\Gamma}\dots\Gamma} &= \frac{1}{(4\pi)^2(d-4)} \int d^4 x \text{tr} [\bar{\alpha} \cdot \gamma \gamma_\mu \nabla_\mu \gamma \cdot \alpha \\ &- 2\bar{\alpha} \cdot \gamma m \gamma \cdot \alpha + 2\bar{\alpha} \cdot \gamma \beta Y - 2\bar{\beta} \gamma \cdot \alpha Y \\ &- \bar{\alpha} \cdot \gamma \gamma_\mu \gamma_\nu \nabla_\mu \nabla_\nu \beta + \bar{\beta} \gamma_\mu \gamma_\nu \nabla_\mu \nabla_\nu \gamma \cdot \alpha \\ &- \bar{\alpha} \cdot \gamma (\not{X} m) \beta + \bar{\alpha} \cdot \gamma m \beta \gamma_\mu \nabla_\mu \beta \\ &- 2\bar{\beta} (\not{X} m) \gamma \cdot \alpha - \bar{\beta} m \gamma_\mu \nabla_\mu \gamma \cdot \alpha \\ &+ 2\bar{\alpha} \cdot \gamma m^2 \beta - 2\bar{\beta} m^2 \gamma \cdot \alpha]. \end{aligned} \quad (3.33)$$

Now we insert the explicit expressions for X_μ , Y , f_μ , α_μ , β , m . Discarding again all terms which are $\mathcal{O}(G_F^n)$ with $n \geq 2$, we get

$$\begin{aligned} W_{L=1}^{\text{div}}|_{\bar{\Gamma}\dots\Gamma} &= -\frac{e^2}{(4\pi)^2(d-4)} \\ &\times \int d^4 x \sum_\ell \{ 2\bar{\ell}(\not{\partial} - ie\mathcal{A})\ell + 8m_\ell \bar{\ell} \ell \\ &+ \frac{3F_0^2}{2} \langle (\bar{\ell} \gamma_\mu \nu_{\ell L} \mathcal{Q}_L^w - \bar{\nu}_{\ell L} \gamma_\mu \ell \mathcal{Q}_L^{w\dagger}) \nabla_\mu (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \\ &+ iF_0^2 \langle (\bar{\ell} \gamma_\mu \nu_{\ell L} \mathcal{Q}_L^w - \bar{\nu}_{\ell L} \gamma_\mu \ell \mathcal{Q}_L^{w\dagger}) [u_\mu, \mathcal{Q}_L^{\text{em}}] \rangle \\ &+ 3F_0^2 m_\ell \langle (\bar{\ell} \nu_{\ell L} \mathcal{Q}_L^w + \bar{\nu}_{\ell L} \ell \mathcal{Q}_L^{w\dagger}) \mathcal{Q}_R^{\text{em}} \rangle \}. \end{aligned} \quad (3.34)$$

Transforming back to Minkowski space, our result for the divergent part of the mixed one-loop functional takes the final form

$$\begin{aligned} W_{L=1}^{\text{div}}|_{\bar{\Gamma}\dots\Gamma} &= \frac{e^2}{(4\pi)^2(d-4)} \int d^4 x \\ &\times \sum_\ell \{ -2\bar{\ell}(i\not{\partial} + e\mathcal{A})\ell + 8m_\ell \bar{\ell} \ell \\ &- \frac{3iF_0^2}{2} \langle (\bar{\ell} \gamma_\mu \nu_{\ell L} \mathcal{Q}_L^w - \bar{\nu}_{\ell L} \gamma_\mu \ell \mathcal{Q}_L^{w\dagger}) \nabla^\mu (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \\ &+ F_0^2 \langle (\bar{\ell} \gamma_\mu \nu_{\ell L} \mathcal{Q}_L^w - \bar{\nu}_{\ell L} \gamma_\mu \ell \mathcal{Q}_L^{w\dagger}) [u^\mu, \mathcal{Q}_L^{\text{em}}] \rangle \\ &+ 3F_0^2 m_\ell \langle (\bar{\ell} \nu_{\ell L} \mathcal{Q}_L^w + \bar{\nu}_{\ell L} \ell \mathcal{Q}_L^{w\dagger}) \mathcal{Q}_R^{\text{em}} \rangle \}. \end{aligned} \quad (3.35)$$

4 The next-to-leading order Lagrangian

We are now in the position to construct the most general local action at next-to-leading order which will also renormalize the one-loop divergences discussed in the previous section.

In the absence of virtual photons and leptons, the local action of $\mathcal{O}(p^4)$ is generated by the well-known Gasser–Leutwyler Lagrangian \mathcal{L}_{p^4} [3]:

$$\begin{aligned} \mathcal{L}_{p^4} &= L_1 \langle u_\mu u^\mu \rangle^2 + L_2 \langle u_\mu u^\nu \rangle \langle u^\mu u_\nu \rangle \\ &+ L_3 \langle u_\mu u^\mu u_\nu u^\nu \rangle + L_4 \langle u_\mu u^\mu \rangle \langle \chi_+ \rangle \\ &+ L_5 \langle u_\mu u^\mu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 \\ &+ \frac{1}{4} (2L_8 + L_{12}) \langle \chi_+^2 \rangle + \frac{1}{4} (2L_8 - L_{12}) \langle \chi_-^2 \rangle \end{aligned}$$

$$\begin{aligned}
& -iL_9 \langle f_+^{\mu\nu} u_\mu u_\nu \rangle + \frac{1}{4} (L_{10} + 2L_{11}) \langle f_{+\mu\nu} f_+^{\mu\nu} \rangle \\
& - \frac{1}{4} (L_{10} - 2L_{11}) \langle f_{-\mu\nu} f_-^{\mu\nu} \rangle, \quad (4.1)
\end{aligned}$$

with

$$\begin{aligned}
f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, \\
F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]. \quad (4.2)
\end{aligned}$$

If also the photon is treated as a dynamical degree of freedom, the following local counterterms of $\mathcal{O}(e^2 p^2)$ have to be added [15]:

$$\begin{aligned}
\mathcal{L}_{e^2 p^2} &= e^2 F_0^2 \left\{ \frac{1}{2} K_1 \langle (\mathcal{Q}_L^{\text{em}})^2 + (\mathcal{Q}_R^{\text{em}})^2 \rangle \langle u_\mu u^\mu \rangle \right. \\
&+ K_2 \langle \mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} \rangle \langle u_\mu u^\mu \rangle \\
&- K_3 [\langle \mathcal{Q}_L^{\text{em}} u_\mu \rangle \langle \mathcal{Q}_L^{\text{em}} u^\mu \rangle + \langle \mathcal{Q}_R^{\text{em}} u_\mu \rangle \langle \mathcal{Q}_R^{\text{em}} u^\mu \rangle] \\
&+ K_4 \langle \mathcal{Q}_L^{\text{em}} u_\mu \rangle \langle \mathcal{Q}_R^{\text{em}} u^\mu \rangle \\
&+ K_5 \langle [(\mathcal{Q}_L^{\text{em}})^2 + (\mathcal{Q}_R^{\text{em}})^2] u_\mu u^\mu \rangle \\
&+ K_6 \langle (\mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} + \mathcal{Q}_R^{\text{em}} \mathcal{Q}_L^{\text{em}}) u_\mu u^\mu \rangle \\
&+ \frac{1}{2} K_7 \langle (\mathcal{Q}_L^{\text{em}})^2 + (\mathcal{Q}_R^{\text{em}})^2 \rangle \langle \chi_+ \rangle \\
&+ K_8 \langle \mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} \rangle \langle \chi_+ \rangle \\
&+ K_9 \langle [(\mathcal{Q}_L^{\text{em}})^2 + (\mathcal{Q}_R^{\text{em}})^2] \chi_+ \rangle \\
&+ K_{10} \langle (\mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} + \mathcal{Q}_R^{\text{em}} \mathcal{Q}_L^{\text{em}}) \chi_+ \rangle \\
&- K_{11} \langle (\mathcal{Q}_L^{\text{em}} \mathcal{Q}_R^{\text{em}} - \mathcal{Q}_R^{\text{em}} \mathcal{Q}_L^{\text{em}}) \chi_- \rangle \\
&- iK_{12} \langle [(\widehat{\nabla}_\mu \mathcal{Q}_L^{\text{em}}) \mathcal{Q}_L^{\text{em}} - \mathcal{Q}_L^{\text{em}} \widehat{\nabla}_\mu \mathcal{Q}_L^{\text{em}} \\
&- (\widehat{\nabla}_\mu \mathcal{Q}_R^{\text{em}}) \mathcal{Q}_R^{\text{em}} + \mathcal{Q}_R^{\text{em}} \widehat{\nabla}_\mu \mathcal{Q}_R^{\text{em}}] u^\mu \rangle \\
&+ K_{13} \langle (\widehat{\nabla}_\mu \mathcal{Q}_L^{\text{em}}) (\widehat{\nabla}^\mu \mathcal{Q}_R^{\text{em}}) \rangle \\
&+ K_{14} \langle (\widehat{\nabla}_\mu \mathcal{Q}_L^{\text{em}}) (\widehat{\nabla}^\mu \mathcal{Q}_L^{\text{em}}) \rangle \\
&+ (\widehat{\nabla}_\mu \mathcal{Q}_R^{\text{em}}) (\widehat{\nabla}^\mu \mathcal{Q}_R^{\text{em}}) \left. \right\}, \quad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\nabla}_\mu \mathcal{Q}_L^{\text{em}} &= \nabla_\mu \mathcal{Q}_L^{\text{em}} + \frac{i}{2} [u_\mu, \mathcal{Q}_L^{\text{em}}] = u (D_\mu \mathcal{Q}_L^{\text{em}}) u^\dagger, \\
\widehat{\nabla}_\mu \mathcal{Q}_R^{\text{em}} &= \nabla_\mu \mathcal{Q}_R^{\text{em}} - \frac{i}{2} [u_\mu, \mathcal{Q}_R^{\text{em}}] = u^\dagger (D_\mu \mathcal{Q}_R^{\text{em}}) u, \quad (4.4)
\end{aligned}$$

with

$$\begin{aligned}
D_\mu \mathcal{Q}_L^{\text{em}} &= \partial_\mu \mathcal{Q}_L^{\text{em}} - i[l_\mu, \mathcal{Q}_L^{\text{em}}], \\
D_\mu \mathcal{Q}_R^{\text{em}} &= \partial_\mu \mathcal{Q}_R^{\text{em}} - i[r_\mu, \mathcal{Q}_R^{\text{em}}]. \quad (4.5)
\end{aligned}$$

In the presence of virtual leptons, the structure of the Lagrangians (4.1) and (4.3) remains unchanged. The only necessary modification is the inclusion of the lepton term in l_μ (see (2.3)). In addition to \mathcal{L}_{p^4} and $\mathcal{L}_{e^2 p^2}$, we need the ‘‘leptonic’’ Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{lept}} &= e^2 \sum_\ell \left\{ F_0^2 [X_1 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle u^\mu \{ \mathcal{Q}_R^{\text{em}}, \mathcal{Q}_L^{\text{w}} \} \rangle \right. \\
&+ X_2 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle u^\mu [\mathcal{Q}_R^{\text{em}}, \mathcal{Q}_L^{\text{w}}] \rangle
\end{aligned}$$

$$\begin{aligned}
&+ X_3 m_\ell \bar{\ell} \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \mathcal{Q}_R^{\text{em}} \rangle \\
&+ iX_4 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \widehat{\nabla}^\mu \mathcal{Q}_L^{\text{em}} \rangle \\
&+ iX_5 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle \mathcal{Q}_L^{\text{w}} \widehat{\nabla}^\mu \mathcal{Q}_R^{\text{em}} \rangle + \text{h.c.} \left. \right] \\
&+ X_6 \bar{\ell} (i \not{\partial} + e \not{A}) \ell \\
&+ X_7 m_\ell \bar{\ell} \ell \}. \quad (4.6)
\end{aligned}$$

In $\mathcal{L}_{\text{lept}}$ we consider only terms quadratic in the lepton fields and at most linear in G_F . The coupling constants X_1, \dots, X_5 are real in the limit of CP invariance and the reality of X_6 and X_7 is required by the hermiticity of the associated action. The terms with $X_{4,5}$ will not appear in realistic physical processes as the generated amplitudes contain an external (axial-) vector source (see (4.4) and (4.5)).

In deriving a minimal set of terms in (4.6), we have used partial integration, the equations of motion (3.4) and the following relations derived from (2.5), (2.7) and (2.8):

$$\mathcal{Q}_L^{\text{em}} \mathcal{Q}_L^{\text{w}} = \frac{2}{3} \mathcal{Q}_L^{\text{w}}, \quad \mathcal{Q}_L^{\text{w}} \mathcal{Q}_L^{\text{em}} = -\frac{1}{3} \mathcal{Q}_L^{\text{w}}, \quad \langle \mathcal{Q}_L^{\text{w}} \rangle = 0. \quad (4.7)$$

We give here some typical examples of terms which can be eliminated in this way:

$$\begin{aligned}
e^2 F_0^2 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle u^\mu [\mathcal{Q}_L^{\text{em}}, \mathcal{Q}_L^{\text{w}}] \rangle &= e^2 F_0^2 \bar{\ell} \gamma_\mu \nu_{\ell L} \langle u^\mu \mathcal{Q}_L^{\text{w}} \rangle \\
&\rightarrow 2e^2 \bar{\ell} (i \not{\partial} + e \not{A} - m_\ell) \ell, \\
e^2 \bar{\nu}_{\ell L} i \not{\partial} \nu_{\ell L} &\rightarrow e^2 \bar{\ell} (i \not{\partial} + e \not{A} - m_\ell) \ell. \quad (4.8)
\end{aligned}$$

Finally, also a photon Lagrangian

$$\mathcal{L}_\gamma = e^2 X_8 F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (4.9)$$

has to be added. This term cancels the divergences of the photon two-point function generated by the lepton loops. (The loop contributions of the charged pseudoscalars are renormalized by the $L_{10,11}$ terms in (4.1).)

The low-energy couplings L_i, K_i, X_i arising here are divergent (except $L_3, L_7, K_7, K_{13}, K_{14}$ and X_1). They absorb the divergences of the one-loop graphs via the renormalization

$$\begin{aligned}
L_i &= L_i^r(\mu) + \Gamma_i \Lambda(\mu), \quad i = 1, \dots, 12, \\
K_i &= K_i^r(\mu) + \Sigma_i \Lambda(\mu), \quad i = 1, \dots, 14, \\
X_i &= X_i^r(\mu) + \Xi_i \Lambda(\mu), \quad i = 1, \dots, 8, \\
\Lambda(\mu) &= \frac{\mu^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2} [\ln(4\pi) + \Gamma'(1) + 1] \right\}, \quad (4.10)
\end{aligned}$$

in the dimensional regularization scheme. The coefficients Γ_i and Σ_i can be found in [3] and in [15], respectively. Their values are not modified by the presence of virtual leptons as long as contributions of $\mathcal{O}(G_F^2)$ are neglected.

The ‘‘new’’ coefficients Ξ_1, \dots, Ξ_7 are determined by (3.35) and Ξ_8 is derived from the divergent part of the lepton loops:

$$\begin{aligned}
\Xi_1 &= 0, \quad \Xi_2 = -\frac{3}{4}, \quad \Xi_3 = -3, \quad \Xi_4 = -\frac{3}{2}, \\
\Xi_5 &= \frac{3}{2}, \quad \Xi_6 = -5, \quad \Xi_7 = -1, \quad \Xi_8 = -\frac{4}{3}. \quad (4.11)
\end{aligned}$$

5 $\pi \rightarrow \ell\nu_\ell$ and $K \rightarrow \ell\nu_\ell$

We are now in the position to perform a complete one-loop analysis of semileptonic pion and kaon decays including the electromagnetic contributions of $\mathcal{O}(e^2 p^2)$. As an illustration we give here the theoretical results for the decay rates of $\pi \rightarrow \ell\nu_\ell$ and $K \rightarrow \ell\nu_\ell$. The former reaction will also serve as the reference process for our further investigations of semileptonic decays.

The contributions of graphs without virtual leptons to the $P \rightarrow \ell\nu_\ell$ amplitudes ($P = \pi, K$) have already been given in (5.3) and (5.4) of [17]. Now we have to add also those diagrams in which a virtual photon is attached to a charged lepton line, the contributions generated by the counterterms in (4.6), and the leptonic wave function and mass renormalization. The decay amplitude obtained in this way is, of course, UV finite but still IR divergent. To arrive at an infrared-finite result, we consider the observable $\Gamma(P \rightarrow \ell\nu_\ell(\gamma)) := \Gamma(P \rightarrow \ell\nu_\ell) + \Gamma(P \rightarrow \ell\nu_\ell\gamma)$. The relevant expression for the total probability of inner bremsstrahlung can be found in (4) of [24]. (We have checked this formula.)

Combining the various contributions we obtain our final result for the $\pi_{\ell 2(\gamma)}$ decay:

$$\begin{aligned} \Gamma(\pi \rightarrow \ell\nu_\ell(\gamma)) &= \frac{G_F^2 |V_{ud}|^2 F_0^2 m_\ell^2 M_{\pi^\pm}}{4\pi} (1 - z_{\pi\ell})^2 \\ &\times \left\{ 1 + \frac{8}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\pi^2] \right. \\ &- \frac{1}{2(4\pi)^2 F_0^2} \left[2M_{\pi^\pm}^2 \ln \frac{M_{\pi^\pm}^2}{\mu^2} + 2M_{\pi^0}^2 \ln \frac{M_{\pi^0}^2}{\mu^2} \right. \\ &+ M_{K^\pm}^2 \ln \frac{M_{K^\pm}^2}{\mu^2} + M_{K^0}^2 \ln \frac{M_{K^0}^2}{\mu^2} \left. \right] \\ &\left. + e^2 E^r(\mu) + \frac{e^2}{(4\pi)^2} \left[3 \ln \frac{M_\pi^2}{\mu^2} + H(z_{\pi\ell}) \right] \right\}, \end{aligned} \quad (5.1)$$

with $z_{\pi\ell} = m_\ell^2/M_{\pi^\pm}^2$, where m_ℓ denotes now the physical mass of the charged lepton. The parameter $E^r(\mu)$ denotes the finite (renormalized) part of the linear combination of coupling constants

$$\begin{aligned} E := & \frac{8}{3}K_1 + \frac{8}{3}K_2 + \frac{20}{9}K_5 + \frac{20}{9}K_6 + 4K_{12} \\ & - \frac{4}{3}X_1 - 4X_2 + 4X_3 - X_6. \end{aligned} \quad (5.2)$$

The kinematical function $H(z)$ is given by

$$\begin{aligned} H(z) = & \frac{23}{2} - \frac{3}{1-z} + 11 \ln z - \frac{2 \ln z}{1-z} - \frac{3 \ln z}{(1-z)^2} \\ & - 8 \ln(1-z) - \frac{4(1+z)}{1-z} \ln z \ln(1-z) \\ & + \frac{8(1+z)}{1-z} \int_0^{1-z} dt \frac{\ln(1-t)}{t}. \end{aligned} \quad (5.3)$$

The scale independence of (5.1) can be checked by using the pertinent β function of the coupling (5.2) and the

(lowest-order) expressions for the pseudoscalar masses

$$\begin{aligned} M_{\pi^\pm}^2 &= 2B_0\hat{m} + 2e^2 Z F_0^2, \\ M_{\pi^0}^2 &= 2B_0\hat{m}, \\ M_{K^\pm}^2 &= B_0 \left[(m_s + \hat{m}) - \frac{2\varepsilon}{\sqrt{3}}(m_s - \hat{m}) \right] + 2e^2 Z F_0^2, \\ M_{K^0}^2 &= B_0 \left[(m_s + \hat{m}) + \frac{2\varepsilon}{\sqrt{3}}(m_s - \hat{m}) \right], \\ M_\eta^2 &= \frac{4}{3}B_0 \left(m_s + \frac{\hat{m}}{2} \right). \end{aligned} \quad (5.4)$$

The mixing angle ε is given by

$$\varepsilon = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}}, \quad (5.5)$$

the symbol \hat{m} stands for the mean value of the light quark masses,

$$\hat{m} = \frac{1}{2}(m_u + m_d), \quad (5.6)$$

and B_0 is the vacuum condensate parameter contained in χ_+ . Finally, M_π and M_K in (5.1) denote the isospin limits ($m_u = m_d, e = 0$) of the pion mass and the kaon mass, respectively:

$$M_\pi^2 = 2B_0\hat{m}, \quad M_K^2 = B_0(m_s + \hat{m}). \quad (5.7)$$

Equation (5.1) allows in principle to extract the value of the pion decay constant F_π from the experimental knowledge of the $\pi_{\ell 2(\gamma)}$ decay rate. In order to obtain an unambiguous answer, let us define F_π as being given by the matrix element of the appropriate axial current between the vacuum and the charged pion state in pure QCD. At one loop, this gives the following (scale independent) expression [3]

$$\begin{aligned} F_\pi = F_0 \left\{ 1 + \frac{4}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\pi^2] \right. \\ \left. - \frac{1}{2(4\pi)^2 F_0^2} \left[2M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} + M_K^2 \ln \frac{M_K^2}{\mu^2} \right] \right\}. \end{aligned} \quad (5.8)$$

Extensive studies of the $\mathcal{O}(e^2)$ radiative corrections to $\pi_{\ell 2}$ decays already exist in the literature [24–26]. Marciano and Sirlin [26] have summarized the presently known short- and long-distance radiative corrections to $\Gamma(\pi \rightarrow \ell\nu_\ell(\gamma))$. Their result, given in (7a) of [26], agrees with the general structure of (5.1). Note that the kinematical function $F(x)$ defined in (7b) of [26] is related to (5.3) by $F(x) = H(x^2)/4 - 1/2$, and that the terms associated with the parameters C_2, C_3 of [26] are beyond the scope of our next-to-leading order analysis. In order to make contact with [26], we need to specify the physical meaning we ascribe to the constant G_F , which so far has mainly served as a book-keeping device. From the point of view of the low-energy approach we are following, it seems natural to identify G_F with the muon decay constant. To lowest order in the weak interactions, the muon decay amplitude follows from the effective Lagrangian

$$\mathcal{L}'_{\text{lept}} = -\frac{G_F}{\sqrt{2}} [\bar{\mu}\gamma_\mu(1 - \gamma_5)\nu_\mu] [\bar{e}\gamma^\mu(1 - \gamma_5)\nu_e]. \quad (5.9)$$

When radiative corrections are included, the total muon decay rate to leading order in G_F remains finite order by order in powers of the fine structure constant α [27]. At order α [28],

$$\Gamma(\mu \rightarrow \text{all}) = \frac{G_F^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right) \left[1 + \frac{\alpha}{2\pi} \left(\frac{25}{4} - \pi^2\right) + \mathcal{O}(\alpha^2)\right], \quad (5.10)$$

with $f(x) = -8x - 12x^2 \ln x + 8x^3 - x^4$ for massless neutrinos. The corrections of order α^2 have recently been computed in [29] (see also the discussion in [30]). Up to a negligible correction factor $1 + 3m_\mu^2/10M_W^2$, G_F can be identified with the constant G_μ appearing in (7a) of [26]. If we further identify $f_\pi/\sqrt{2}$ occurring in the same expression with the QCD quantity F_π as defined above, we obtain for the constant C_1 of [26]

$$C_1 = -4\pi^2 E^r(M_\rho) - \frac{1}{2} + \ln \frac{M_Z^2}{M_\rho^2} + \frac{Z}{4} \left[3 + 2 \ln \frac{M_\pi^2}{M_\rho^2} + \ln \frac{M_K^2}{M_\rho^2}\right]. \quad (5.11)$$

Quite analogously, we find for the $K_{\ell 2(\gamma)}$ decay:

$$\begin{aligned} \Gamma(K \rightarrow \ell \nu_\ell(\gamma)) &= \frac{G_F^2 |V_{us}|^2 F_0^2 m_\ell^2 M_{K^\pm} (1 - z_{K\ell})^2}{4\pi} \\ &\times \left\{ 1 + \frac{8}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_K^2] \right. \\ &- \frac{1}{4(4\pi)^2 F_0^2} \left[2M_{\pi^\pm}^2 \ln \frac{M_{\pi^\pm}^2}{\mu^2} + M_{\pi^0}^2 \ln \frac{M_{\pi^0}^2}{\mu^2} \right. \\ &+ 4M_{K^\pm}^2 \ln \frac{M_{K^\pm}^2}{\mu^2} + 2M_{K^0}^2 \ln \frac{M_{K^0}^2}{\mu^2} + 3M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \left. \right] \\ &+ \frac{16\sqrt{3}\varepsilon}{3F_0^2} L_5^r(\mu)(M_\pi^2 - M_K^2) \\ &- \frac{\sqrt{3}\varepsilon}{2(4\pi)^2 F_0^2} \left[M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \right] \\ &\left. + e^2 E^r(\mu) + \frac{e^2}{(4\pi)^2} \left[3 \ln \frac{M_K^2}{\mu^2} + H(z_{K\ell}) \right] \right\}, \quad (5.12) \end{aligned}$$

with $z_{K\ell} = m_\ell^2/M_{K^\pm}^2$. Note that (5.1) as well as (5.12) contain the same (unknown) electromagnetic low-energy coupling constant $E^r(\mu)$. Taking the ratio of (5.12) and (5.1), $E^r(\mu)$ cancels and therefore the electromagnetic corrections to this specific combination of observables are uniquely determined at the order we consider. In principle, this result could be used for an improvement in extracting the (strong) low-energy coupling L_5 from the experimental data. In practice, however, the uncertainties due to higher-order strong interaction effects are much bigger than the electromagnetic corrections. Nevertheless, this example shows that certain observables allow for unambiguous predictions of the associated electromagnetic contributions in spite of our (presently) poor knowledge about the values of the electromagnetic low-energy couplings. The same is, of course, also true for the ratios $\Gamma(P \rightarrow e \nu_e(\gamma))\Gamma(P \rightarrow \mu \nu_\mu(\gamma))$.

Defining the decay constant of the charged kaon as described above for the pion¹ we obtain [3]

$$\begin{aligned} F_{K^\pm} &= F_0 \left\{ 1 + \frac{4}{F_0^2} [L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_K^2] \right. \\ &- \frac{1}{8(4\pi)^2 F_0^2} \left[3M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} + 6M_K^2 \ln \frac{M_K^2}{\mu^2} \right. \\ &+ 3M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \left. \right] - \frac{8\sqrt{3}\varepsilon}{3F_0^2} L_5^r(\mu)(M_K^2 - M_\pi^2) \\ &- \frac{\sqrt{3}\varepsilon}{4(4\pi)^2 F_0^2} \left[M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \right. \\ &\left. \left. - \frac{2}{3}(M_K^2 - M_\pi^2) \left(\ln \frac{M_K^2}{\mu^2} + 1 \right) \right] \right\}. \quad (5.13) \end{aligned}$$

Performing the replacements $M_{\pi^\pm} \rightarrow M_{K^\pm}$ and $V_{ud} \rightarrow V_{us}$ in (7a) of [26], we find that in the case of the kaon decay the constant C_1 corresponds to

$$C_1 = -4\pi^2 E^r(M_\rho) - \frac{1}{2} + \ln \frac{M_Z^2}{M_\rho^2} + \frac{Z}{4} \left[3 + \ln \frac{M_\pi^2}{M_\rho^2} + 2 \ln \frac{M_K^2}{M_\rho^2} \right]. \quad (5.14)$$

6 Conclusions

We have developed the appropriate low-energy effective theory for a complete treatment of isospin-violating effects in semileptonic weak processes. The electromagnetic interaction requires the inclusion of the photon and the light lepton fields as explicit dynamical degrees of freedom in the chiral Lagrangian. At next-to-leading order, the list of local terms given by Gasser and Leutwyler [3] for the QCD part and by Urech [15] for the electromagnetic interaction of the pseudoscalars has to be enlarged. This is, of course, a consequence of the presence of virtual leptons in our extended theory. Regarding pure lepton or photon bilinears as “trivial”, five additional “non-trivial” terms of this type are arising. Two of them will, however, not appear in realistic physical processes. One may therefore conclude that the main bulk of electromagnetic low-energy constants is already contained in Urech’s Lagrangian and the inclusion of virtual leptons in chiral perturbation theory does not substantially aggravate the problem of unknown parameters.

The continuation of the present work will follow two principal lines. Firstly, we are now in the position to calculate the electromagnetic corrections to semileptonic weak

¹ In the case of the pion, the distinction between the charged and the neutral decay constant is a tiny $\mathcal{O}((m_d - m_u)^2)$ effect, and arises only at higher orders, whereas F_{K^\pm}/F_{K^0} is $\mathcal{O}((m_d - m_u))$.

decays where all constraints imposed by chiral symmetry are taken into account. In spite of our large ignorance of the actual values of the electromagnetic low-energy couplings, it will often be possible to relate the electromagnetic contributions to different processes in this way. For specific combinations of observables one might even find parameter-free predictions. Simple examples of this kind have been given for the $P_{\ell 2}$ decays. In some fortunate cases simple order-of-magnitude estimates for the electromagnetic couplings based on chiral dimensional analysis may even be sufficient. (See for instance [16,17].)

The second major task for the near future is, of course, the determination of the physical values of the coupling constants K_i^r and X_i^r in the standard model. In contrast to the rather good information on the QCD effective couplings L_1^r, \dots, L_{10}^r that can be obtained in the standard framework [2,3], only very little is known so far in the electromagnetic sector. First attempts to estimate some of the K_i can be found in [31–33]. As far as the constants X_i are concerned, the recent analysis [34] of the counterterms contributing to the decay processes of light neutral pseudoscalars into charged lepton pairs leads one to expect that reliable estimates for these constants can be achieved within a large- N_c approach. Only with a more precise information about the K_i^r and X_i^r will the electromagnetic part of the chiral effective theory finally exhibit its full predictive power.

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Appendix

The bosonic covariant derivative occurring in (3.9),

$$D_\rho = \partial_\rho + X_\rho \quad (\text{A.1})$$

has the index structure

$$X_\rho = -X_\rho^T = \begin{bmatrix} (X_\rho)_{ij} & (X_\rho)_{i\nu} \\ (X_\rho)_{\mu j} & (X_\rho)_{\mu\nu} \end{bmatrix}, \quad (\text{A.2})$$

where the matrix elements are given by

$$\begin{aligned} (X_\rho)_{ij} &= -\frac{1}{2} \langle T_\rho [\lambda_i, \lambda_j] \rangle, \\ (X_\rho)_{i\nu} &= \frac{eF_0}{4} \delta_{\rho\nu} \langle (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \lambda_i \rangle, \\ (X_\rho)_{\mu j} &= -(X_\rho)_{j\mu}, \\ (X_\rho)_{\mu\nu} &= 0. \end{aligned} \quad (\text{A.3})$$

The bosonic field $Y = Y^T$ has the same index structure as (A.2) with

$$Y_{ij} = s_{ij} - \frac{e^2 F_0^2}{4} \langle \lambda_i (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \langle \lambda_j (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle,$$

$$\begin{aligned} Y_{i\nu} &= -\frac{eF_0}{4} \langle \lambda_i \{ \nabla_\nu (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) + i[u_\nu, \mathcal{Q}_R^{\text{em}} + \mathcal{Q}_L^{\text{em}}] \} \rangle, \\ Y_{\mu j} &= Y_{j\mu}, \\ Y_{\mu\nu} &= \frac{3e^2 F_0^2}{8} \delta_{\mu\nu} \langle (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}})^2 \rangle. \end{aligned} \quad (\text{A.4})$$

The covariant derivative acting on the fermions,

$$\mathcal{D}_\rho = \partial_\rho + i f_\rho \quad (\text{A.5})$$

is determined by

$$f_\rho = \begin{bmatrix} (f_\rho)_{ab} & (f_\rho)_{an} \\ (f_\rho)_{mb} & (f_\rho)_{mn} \end{bmatrix} \quad (\text{A.6})$$

with

$$\begin{aligned} (f_\rho)_{ab} &= -eA_\rho \delta_{ab}, \\ (f_\rho)_{an} &= \frac{F_0^2}{2} \langle u_\rho \mathcal{Q}_L^{\text{w}} \rangle \delta_{an}, \\ (f_\rho)_{mb} &= \frac{F_0^2}{2} \langle u_\rho \mathcal{Q}_L^{\text{w}\dagger} \rangle \delta_{mb}, \\ (f_\rho)_{mn} &= 0. \end{aligned} \quad (\text{A.7})$$

The fermion mass matrix m shares the index structure with (A.6). The matrix elements read

$$m_{ab} = m_a \delta_{ab}, \quad m_{an} = m_{mb} = m_{mn} = 0. \quad (\text{A.8})$$

The fields $\alpha_\rho, \bar{\alpha}_\rho, \beta, \bar{\beta}$ are fermionic quantities. α_ρ is given by

$$\alpha_\rho = \begin{bmatrix} (\alpha_\rho)_{aj} & (\alpha_\rho)_{a\nu} \\ (\alpha_\rho)_{mj} & (\alpha_\rho)_{m\nu} \end{bmatrix} \quad (\text{A.9})$$

with

$$\begin{aligned} (\alpha_\rho)_{aj} &= -\frac{iF_0}{4} \langle [u_\rho, \mathcal{Q}_L^{\text{w}}] \lambda_j \rangle \nu_{aL}, \\ (\alpha_\rho)_{a\nu} &= -\frac{eF_0^2}{4} \delta_{\rho\nu} \langle \mathcal{Q}_L^{\text{w}} (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \nu_{aL} - e\delta_{\rho\nu} \ell_a, \\ (\alpha_\rho)_{mj} &= -\frac{iF_0}{4} \langle [u_\rho, \mathcal{Q}_L^{\text{w}\dagger}] \lambda_j \rangle \ell_{mL}, \\ (\alpha_\rho)_{m\nu} &= -\frac{eF_0^2}{4} \delta_{\rho\nu} \langle \mathcal{Q}_L^{\text{w}\dagger} (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \ell_{mL}. \end{aligned} \quad (\text{A.10})$$

$\bar{\alpha}_\rho$ takes the form

$$\bar{\alpha}_\rho = \begin{bmatrix} (\bar{\alpha}_\rho)_{ib} & (\bar{\alpha}_\rho)_{in} \\ (\bar{\alpha}_\rho)_{\mu b} & (\bar{\alpha}_\rho)_{\mu n} \end{bmatrix} \quad (\text{A.11})$$

with

$$\begin{aligned} (\bar{\alpha}_\rho)_{ib} &= -\frac{iF_0}{4} \langle [u_\rho, \mathcal{Q}_L^{\text{w}\dagger}] \lambda_i \rangle \bar{\nu}_{bL}, \\ (\bar{\alpha}_\rho)_{in} &= -\frac{iF_0}{4} \langle [u_\rho, \mathcal{Q}_L^{\text{w}}] \lambda_i \rangle \bar{\ell}_{nL}, \\ (\bar{\alpha}_\rho)_{\mu b} &= -\frac{eF_0^2}{4} \delta_{\rho\mu} \langle \mathcal{Q}_L^{\text{w}\dagger} (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \bar{\nu}_{bL} - e\delta_{\rho\mu} \bar{\ell}_b, \\ (\bar{\alpha}_\rho)_{\mu n} &= -\frac{eF_0^2}{4} \delta_{\rho\mu} \langle \mathcal{Q}_L^{\text{w}} (\mathcal{Q}_R^{\text{em}} - \mathcal{Q}_L^{\text{em}}) \rangle \bar{\ell}_{nL}. \end{aligned} \quad (\text{A.12})$$

Finally,

$$\begin{aligned}\beta_{aj} &= -\frac{F_0}{2}\langle Q_L^w \lambda_j \rangle \nu_{aL}, \\ \beta_{mj} &= -\frac{F_0}{2}\langle Q_L^{w\dagger} \lambda_j \rangle \ell_{mL}, \\ \beta_{av} &= \beta_{mv} = 0,\end{aligned}\tag{A.13}$$

and

$$\begin{aligned}\bar{\beta}_{ib} &= -\frac{F_0}{2}\langle Q_L^{w\dagger} \lambda_i \rangle \bar{\nu}_{bL}, \\ \bar{\beta}_{in} &= -\frac{F_0}{2}\langle Q_L^w \lambda_i \rangle \bar{\ell}_{nL}, \\ \bar{\beta}_{\mu b} &= \bar{\beta}_{\mu n} = 0.\end{aligned}\tag{A.14}$$

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